Lecture 8 (1/21/22).

Recall. $G \subseteq \mathbb{C}$ open, (Ω, d) complete metric space, $\mathcal{C}(G,\Omega)$ space of cont. fcns $f: G \to \Omega$.

- $\{K_n\}_{n=1}^{\infty}$ is an exhaustion of G by compacts: $K_n \subseteq \operatorname{int} K_{n+1}$, $\forall K \subset\subset G$ $\exists N$ s.t. $K \subset\subset K_N$. $\Rightarrow$ $G = \bigcup_{n=1}^{\infty} K_n$.
  (We proved that we may obtain an additional property, but this will not be needed for now.)

depends on $\{K_n\}$

- Set $\rho_n(f,g) = \sup_{K_n} d(f,g) \Rightarrow$ metric:

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

Thm 1. The collection of open sets in $(\mathcal{C}(G,\Omega), \rho)$ does not depend on exhaustion $\{K_n\}_{n=1}^{\infty}$ of G.

Prop 3. (1) $U \subseteq \mathcal{C}(G,\Omega)$ is open $\Leftrightarrow$ $\forall f \in U\ \exists K \subset\subset G$ and $\varepsilon > 0$ s.t. $\{g : \sup_K d(f,g) < \varepsilon\} \subseteq U$.

(2) $\{f_n\}_{n=1}^{\infty}$ converges to f in $\mathcal{C}(G,\Omega) \Leftrightarrow$ $f_n \to f$ unif. on every compact $K \subset\subset G$.

Pf. (1) $\Rightarrow$. Pick $f \in U$. Then, $\exists \delta > 0$ s.t. $\{\rho(f,g) < \delta\} \subseteq U$ since U open. $\exists N$ s.t $\sum_{n=N+1}^{\infty} 2^{-n} < \delta/2$. Let $K = K_N$ and $\varepsilon = \frac{\delta}{2}\left(\sum_{n=1}^{N} 2^{-n}\right)^{-1}$. Then, if $\rho_N(f,g) = \sup_{K=K_N} d(f,g) < \varepsilon$, $\rho_n(f,g) \le \rho_N(f,g) < \varepsilon$ for $n \le N$ (since $K_n \subseteq K_N$) and hence

$$\rho(f,g) = \sum_{n=1}^{N} 2^{-n} \frac{\rho_n(f,g)}{1+\rho_n(f,g)} + \sum_{n=N+1}^{\infty} \text{same}$$

$$< \varepsilon \sum_{1}^{N} 2^{-n} + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} < \delta \quad \Rightarrow$$

$\{\sup_K d(f,g) < \varepsilon\} \subseteq U$.

$\Leftarrow$. Pick $f \in U$, and let K, ε be as in assumption. By exhaustive prop. $\exists N$ s.t. $K \subseteq K_N$. Let $\delta > 0$ s.t. $0 < \frac{2^N \delta}{1-2^N\delta} < \varepsilon$.

If $\rho(f,g) < \delta$, then $2^{-N} \frac{\rho_N(f,g)}{1+\rho_N(f,g)} < \delta$

$\Rightarrow \rho_N(f,g) < \frac{2^N\delta}{1-2^N\delta} < \varepsilon \Rightarrow \sup_K d(f,g) < \varepsilon$

$\Rightarrow g \in U$. This completes pf. of (1). ∎

(2) $\Rightarrow$. Let $f_k \to f$ in $C(G,\Omega)$, and let $K \subset\subset G$, $\varepsilon > 0$. Picking N and $\delta > 0$ as in $\Leftarrow$ in (1), we find $\rho(f,f_k) < \delta \Rightarrow$ $\sup_K d(f,f_k) < \varepsilon \Rightarrow f_k \to f$ unif. on K.

$\Leftarrow$. Pick $\delta > 0$. Let N, ε be as in $\Rightarrow$ of (1), we find that $\sup_K d(f,f_k) < \varepsilon \Rightarrow$

$\rho(f, f_k) < \delta \Rightarrow f_k \to f$ in $\mathcal{C}(G, \Omega)$.

$\boxtimes$

Recall Ω is complete.

<u>Thm 2.</u> $(\mathcal{C}(G,\Omega), \rho)$ is complete.

Pf. Let $\{f_k\}$ be Cauchy sequence in $\mathcal{C}(G,\Omega)$. Choose $K \subset\subset G$. The same arg. as in the pf of Thm 1 (2) "$\Rightarrow$" above $\Rightarrow$ $\{f_k\}$ is a uniform Cauchy seq. in K. Since Ω is complete and any singleton $\{z\}$ is compact $\Rightarrow \exists f: G \to \Omega$ such that $f_k(z) \to f(z)$ pointwise. But then $f_k \to f$ unif. on any K (HW from Ch. II) $\Rightarrow$ f cont., i.e. $f \in \mathcal{C}(G,\Omega)$. By Thm 1 (2), $f_k \to f$ in $\mathcal{C}(G,\Omega)$. $\boxtimes$